ISRAEL JOURNAL OF MATHEMATICS **129** (2002), 373–380

BOUNDARY BEHAVIOUR OF BMO-QC AUTOMORPHISMS

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ABSTRACT

We give necessary and sufficient conditions on the boundary correspondence of BMO-qc mappings of the upper half-plane which reduce to the Ahlfors–Buerling condition in the case of qc automorphisms.

1. Introduction

It is a well-known result of Ahlfors and Beurling that a K-qc automorphism of the upper half-plane extends to the boundary and satisfies the so-called M-condition there [1]. This condition is also sufficient for a homeomorphism of $\mathbb{R}$ onto itself to have a K-qc extension to the upper half-plane.

A BMO-qc map is defined in [8]. We recall the definition. Let $\Omega \subset \mathbb{C}$ be a domain and $f\colon \Omega \to \mathbb{C}$ be an ACL sense-preserving map for which $\mu(z) = f_{\bar{z}}/f_z$ when f is differentiable and $f_z \neq 0$, and $\mu(z) = 0$ otherwise. If the dilatation

$$K_f(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|},$$

at the point z, satisfies $K_f(z) \leq Q(z)$ a.e. for some BMO function $Q\colon \Omega \to \mathbb{R}$, we say that our map f is BMO-qc. If $Q(z) \equiv K$ for some constant K, we get the usual K-qc maps. BMO-qc maps are closely related to mappings which were studied earlier by David [4] and Tukia [9]. Recall that a real-valued function $u \in L^1_{loc}(D)$ is BMO, i.e., of *bounded mean oscillation*, in D if

$$||u||_* = \sup_{B\subset D} \frac{1}{|B|}\int_B |u(z) - u_B|dxdy < \infty,$$

Received June 24, 1999

where the supremum is taken over all disks B in D and

$$u_B = \frac{1}{|B|} \int_B u(z) dx dy.$$

In [8, 10.2], Ryazanov, Srebro and Yakubov ask the above question for BMO-qc maps. More precisely, let f be a BMO-qc mapping of the upper half-plane H onto itself. Then f has a homeomorphic extension F on $\bar{H}$; see [8, Cor. 8.3]. Find necessary conditions on the boundary correspondence $\phi = F_{|\partial H}$. Find sufficient conditions on a homeomorphism ϕ of $\mathbb{R}$ onto itself for the existence of a BMO-qc extension of ϕ to H. We give both necessary and sufficient conditions on the boundary correspondence of BMO-qc mappings of the upper half-plane which reduce to the Ahlfors–Buerling condition in the case of qc automorphisms. Note that, contrary to the qc case, the inverse of a BMO-qc mapping need not be BMO-qc, and thus there is a certain abuse in having the word automorphism in the title.

ACKNOWLEDGEMENT: I thank Professor U. Srebro for the numerous insightful and illuminating discussions during the course of this work. I also thank the Technion, Israel, for its hospitality while part of this work was completed.

2. A necessary condition

We identify $\mathbb{C}$ with $\mathbb{R}^2$ and use them interchangeably. The constants $c, c_1, c_2, \ldots$ are absolute constants.

THEOREM 2.1: *Let $f\colon H \to H$ be a surjective homeomorphism with $K_f(z) \leq Q(z)$ a.e. for $Q \in \mathrm{BMO}(H)$. Let ϕ denote the induced boundary correspondence. Then ϕ satisfies for each $x, t \in \mathbb{R}$,*

$$\frac{1}{c_1 \exp(c_2 Q_B)} \leq \frac{\phi(x+t) - \phi(x)}{\phi(x) - \phi(x-t)} \leq c_1 \exp(c_2 Q_B), \tag{2.1}$$

where $Q_B = \frac{1}{|B|}\int_B Q(z) dm_2$, $B = B_{2t}(x)$ is a ball centered at x of radius $2t$ and m_2 is 2-dimensional Lebesgue measure.

Remark 2.1: Note that when $Q \equiv$ constant, then the above condition is comparable to the Ahlfors–Buerling condition. Further, we may replace Q_B by the Hardy–Littlewood maximal function $MQ((x,0))$ of Q since

$$Q_{B_{2t}(x)} \leq MQ((x,0)).$$

By a result of Bennet, DeVore and Sharpley, [2], if $Q \in$ BMO then either $MQ \equiv \infty$ or $MQ \in$ BMO.

Proof: We first note that it is enough to show the left side inequality of (2.1). Then the right side follows by reflection. More precisely, set

$$g = \tau \circ f \circ \tau, \tag{2.2}$$

where $\tau\colon \mathbb{C} \to \mathbb{C}$ is reflection in the y-axis. So $\tau(x+iy) = -x+iy$. Now g is a sense-preserving BMO-qc automorphism of H with $K_g(z) = K_f(z) \le Q(z)$ a.e., so the left side inequality of (2.1) holds for g. Using the fact that on the real axis $g(-x) = -f(x)$ and applying the left side of (2.1) to g at the points $-x-t, -x, -x+t$ we obtain the right side inequality of (2.1) immediately. Thus all we need to do is show the left side of (2.1).

We use the modulus inequality [8, Sec. 1], see also [7, p. 173]

$$M(f\Gamma) \le \int\int_{\mathbb{C}} Q(z)\rho^2(z)dm_2. \tag{2.3}$$

Here Γ is a path family in H and ρ is admissible with respect to Γ; i.e., ρ is a non-negative Borel function on H with $\int_\gamma \rho(z)|dz| \ge 1$ for all locally rectifiable paths γ in Γ. Further, $M(f\Gamma)$ is the conformal modulus of $f\Gamma$, i.e.,

$$M(f\Gamma) = \inf \int\int_{\mathbb{C}} \rho^{*2}(z)dm_2$$

where the infimum is taken over all functions ρ^* which are admissible for $f\Gamma$.

We shall define the family $f\Gamma \subset H$ and, since f is a homeomorphism, the family Γ is well-defined. Let $x, t \in \mathbb{R}$ be any numbers. We set

$$h = f(x) - f(x-t), \quad k = f(x+t) - f(x). \tag{2.4}$$

Consider the semi-annular region $(B_{h+k/2}(f(x)+k/2) \setminus B_{k/2}(f(x)+k/2)) \cap H$. Let $f\Gamma$ be all paths that lie in this region and join the two segments on the real axis. Then

$$M(f\Gamma) = \frac{1}{\pi}\log\frac{h+k/2}{k/2}.$$

Further,

$$\rho(z) = \frac{1}{t}\chi_{B_{2t}(x)}(z) \tag{2.5}$$

is admissible with respect to Γ, where χ is the indicator function of the set $B_{2t}(x)$. Using these two facts in (2.3) we get that

$$\frac{1}{\pi}\log\frac{h+k/2}{k/2} = M(f\Gamma) \le \int\int_{B_{2t}(x)} Q(z)\frac{1}{t^2}dm.$$

From this, the left side inequality of (2.1) follows immediately. ∎

3. A sufficient condition

THEOREM 3.1: *Let $\phi: \mathbb{R} \to \mathbb{R}$ be a homeomorphism and $Q \in \mathrm{BMO}(\mathbb{R})$. Further, let ϕ satisfy, for each pair of intervals $I, I' \subset \mathbb{R}$ with $I' \subset I$ and $|I'| = |I|/6$,*

$$1 \leq \frac{|\phi(I)|}{|\phi(I')|} \leq c_3 Q_I. \tag{3.1}$$

Then there exists a BMO*-qc extention of ϕ to H; i.e., there exists $\hat{Q} \in \mathrm{BMO}(H)$ and a sense-preserving ACL homeomorphism $f: \bar{H} \to \bar{H}$ such that $f|_{\partial H} = \phi$ and*

$$K_f(x,y) \leq \hat{Q}(x,y), \quad \text{a.e.} \tag{3.2}$$

Furthermore, $||\hat{Q}||_* \leq c||Q||_*$ where c is an absolute constant and $||\hat{Q}||_*$ and $||Q||_*$ are, respectively, the BMO *norms of $\hat{Q}$ and Q.*

Remark 3.1: Unlike the necessary condition, we do not have exponentials in the sufficiency condition. Also, Q_I is computed with a BMO function Q defined only on $\mathbb{R}$ rather than $\mathbb{R}^2$. When $Q \equiv$ constant, (3.1) implies the Ahlfors–Buerling sufficiency condition.

Proof: We construct $f: H \to H$ such that it is piecewise linear on H. We then compute $K_f(x,y)$, which is naturally peicewise constant on H, and check that (3.2) holds for some $\hat{Q} \in \mathrm{BMO}(H)$.

The construction of f is based on Carleson's construction. Consider the unit square $S = [0,1] \times [0,1]$. First we break up S into countably many rectangles $R_{i,j}$.

$$R_{i,j} = \left[\frac{i}{2^{j-1}}, \frac{i+1}{2^{j-1}}\right] \times \left[\frac{1}{2^j}, \frac{1}{2^{j-1}}\right], \tag{3.3}$$

$i = 0, 1, \ldots, 2^{j-1} - 1$, and $j = 1, 2, \ldots$. By translating this by n, $n \in \mathbb{Z}$, we subdivide $\{(x,y) : 0 \leq y \leq 1\}$. Next, by scaling this by 2^m, $m \geq 1$, we may subdivide $\{(x,y) : 0 \leq y \leq 2^m\}$. Thus we have subdivided H into rectangles. We call this family of rectangles $\mathcal{R}$.

We think of $R \in \mathcal{R}$ as being pentagons with the vertices being the four corners and the mid-point of the bottom side. We label these vertices $v^1, \ldots, v^5$ with v^1 being the lower right corner and going counter-clockwise along ∂R. Note that in this notation we have dropped the dependence on R.

On the five vertices of each $R \in \mathcal{R}$ we define f by

$$f(x,y) = (\phi(x), \phi(x+y) - \phi(x-y)). \tag{3.4}$$

We may now define f on all of R in any appropriate piecewise linear way. The easiest is to divide each $R \in \mathcal{R}$ into three 2-cells by joining the top two corners to the mid-point of the bottom two. Thus join v^3 and v^2 to v^5. For each 2-cell in $R \in \mathcal{R}$ we know f on the vertices. We define f on the 2-cell by its unique affine extention via barycentric coordinates.

In what follows it is convenient to write $R \in \mathcal{R}$ as $R = [x_0 - h, x_0 + h] \times [h, 2h]$ where $h = 2^j$ for some $j \in \mathbb{Z}$.

LEMMA 3.1: *The map f defined above is a sense-preserving ACL homeomorphism of $\bar{H}$ onto itself, with $\phi = f_{\partial H}$. Further, on $R \in \mathcal{R}$, the dilatation of f satisfies*

$$K_f(x,y) \le c_4 Q_{I_R}, \tag{3.5}$$

where $I_R = [x_0 - 3h, x_0 + 3h]$.

Proof: For each $R \in \mathcal{R}$ note that $f(v^2)$ and $f(v^1)$ have the same x-coordinate and $f(v^2)$ has a bigger y-coordinate than $f(v^1)$. Similarly, $f(v^3)$ lies directly above $f(v^4)$. Finally, $f(v^5)$ lies below the line segment $[f(v^3), f(v^2)]$. From this it is clear that on each R, the map f is a sense-preserving ACL homeomorphism onto its image. It follows that $f\colon \bar{H} \to \bar{H}$ is a sense-preserving surjective ACL homeomorphism and $f_{|\partial H} = \phi$. Indeed, one only needs to check surjectivity of f. Consider, for instance, the region $\{(x,y) : 0 \le x \le 1, y \ge 0\}$. The image of the slabs $\{0 \le x \le 1, 2^j \le y \le 2^{j+1}\}$, $j \in \mathbb{Z}$ fill out the region $\{\phi(0) \le x \le \phi(1)\}$. From this surjectivity follows.

To compute the dilatation of f on *a* 2-cell, $T \subset R$, we have the following lemma.

LEMMA 3.2: *Let Δ be the standard 2-cell in $\mathbb{C}$ and let $T_0 \subset \mathbb{C}$ be the triangle with vertices $0, 1, w$, where $\mathrm{Im}(w) > 0$ and $\arg(w) \ge \pi/3$. Let $g\colon \Delta \to T_0$ be the unique affine map such that $g(0) = 0$, $g(1) = 1$ and $g(i) = g(w)$. Then*

$$K_g \le \frac{2}{\sin(\pi/3)} \max\left(|w|, \frac{1}{|w|}\right).$$

Proof: Since g is affine we may write it as $g(z) = Az + B\bar{z}$ for some $A, B \in \mathbb{C}$. Now $g(1) = 1$, $g(i) = w$ yield that $A = (1 - iw)/2$ and $B = (1 + iw)/2$. So

$$\mu_g = \frac{B}{A} = \frac{i - w}{i + w}.$$

It follows that

$$K_g = \frac{1+|w|^2}{Im(w)} \leq \frac{2}{\sin(\pi/3)} \max\left(|w|, \frac{1}{|w|}\right). \qquad \blacksquare$$

To apply this to our situation we label the vertices of $f(T)$ by w_1, w_2, w_3 so that the angle at w_1 is $\geq \pi/3$, and as we go along w_1, w_2, w_3 we give a positive orientation to $\partial f(T)$. Now by using $z \mapsto (z-w_1)/(w_2-w_1)$ we map $f(T)$ to T_0 with

$$w = \frac{w_3 - w_1}{w_2 - w_1}.$$

Label the vertices of T by u_1, u_2, u_3 with $u_i = f^{-1}(w_i)$. Now by using $z \mapsto (z-u_1)/(u_2-u_1)$ we map the isoceles triangle T to Δ', where Δ' is necessarily an isoceles triangle, two of whose vertices are 0 and 1 and the third is $i, 1+i$ or $1/2 + i/\sqrt{2}$. Now by uniqueness of the affine map f between T and $f(T)$ we get, by applying Lemma 3.2 twice if Δ' differs from Δ, that on T

$$K_f \leq c \max\left(\frac{|w_3 - w_1|}{|w_2 - w_1|}, \frac{|w_2 - w_1|}{|w_3 - w_1|}\right). \tag{3.6}$$

So on each triangle T the dilatation of f is bounded by a constant times the maximum of the ratio of the length of the sides of the image triangle $f(T)$.

Recall that R contains three triangles T whose sides can be collectively labelled $b_1, \ldots, b_7$. On R, we can now conclude that

$$K_f \leq c \max_{1\leq j,k\leq 7}\left(\frac{|b_j|}{|b_k|}\right). \tag{3.7}$$

To complete the proof of the lemma recall $R = [x_0 - h, x_0 + h] \times [h, 2h]$ as above. Let $I_1 = [x_0 - 3h, x_0 - 2h]$, and $I_i = I_1 + (i-1)h, i = 1, \ldots, 6$. Let $J_i = \phi(I_i)$ and $J_R = \bigcup_1^6 J_i$, $I_R = \bigcup_1^6 I_i$. Then, by the definition of $v^1, \ldots, v^5$ and (3.4),

$$\begin{aligned}
f(v^1) &= (\phi(x_0+h), |J_4| + |J_5|),\\
f(v^2) &= (\phi(x_0+h), |J_3| + |J_4| + |J_5| + |J_6|),\\
f(v^3) &= (\phi(x_0-h), |J_1| + |J_2| + |J_3| + |J_4|),\\
f(v^4) &= (\phi(x_0-h), |J_2| + |J_3|),\\
f(v^5) &= (\phi(x_0), |J_3| + |J_4|).
\end{aligned}$$

One can easily compute b_k from the above for $k = 1, \ldots, 7$. Further, from the sufficiency condition (3.1) we have that for all $i = 1, \ldots, 6$,

$$1 \leq \frac{|J_R|}{|J_i|} \leq c_3 Q_{I_R}.$$

Hence it follows that for all $k = 1, \ldots, 7$,

$$\frac{1}{c_5 Q_{I_R}} \leq \frac{|b_k|}{|J_R|} \leq c_4. \tag{3.8}$$

So on $R \in \mathcal{R}$ we have, by (3.7) and (3.8), that

$$K_f \leq c \max_{1 \leq j,k \leq 7} \left(\frac{|b_j|}{|b_k|} \right) \leq c_6 Q_{I_R}. \tag{3.9}$$

Finally, we define a BMO function $\hat{Q}$ which dominates K_f. Let $\hat{Q}\colon H \to \mathbb{R}$ be such that on $R = [x_0 - h, x_0 + h] \times [h, 2h] \in \mathcal{R}$ we have

$$\hat{Q} \equiv c_6 Q_{I_R} \tag{3.10}$$

where $I_R = [x_0 - 3h, x_0 + 3h]$. ∎

LEMMA 3.3: *The map $\hat{Q}$ defined above is in* BMO(H).

Proof: For each ball $B \subset H$ we will define a constant c_B such that

$$\sup_B \frac{1}{|B|} \int_B |\hat{Q} - c_B| dm_2 = A, \tag{3.11}$$

where A is some constant. This will imply that $\hat{Q}$ is in BMO(H). Indeed, we show that $A = c||Q||_*$ and so $||\hat{Q}||_* = 2A = 2c||Q||_*$.

Let $B \subset H$ be any ball of radius r, say. Let m be the smallest integer such that $B \subset \{(x,y) : 0 < y \leq 2^{m+1}\}$. Clearly $2r \leq 2^{m+1}$. Consider two cases.

CASE (1): $r < 2^{m-2}$. Then $B \subset \{2^{m-1} < y \leq 2^{m+1}\}$. In each slab

$$\{2^m \leq y \leq 2^{m+1}\} \quad \text{and} \quad \{2^{m-1} \leq y \leq 2^m\},$$

B meets at most two rectangles R because $2r < 2^{m-1}$ and the width of the rectangles is 2^m or more. Let $R' \subset \{2^m \leq y \leq 2^{m+1}\}$ be such that $R' \cap B \neq \emptyset$.

CASE (2): $2^{m-2} \leq r$. Then $2^{m-2} \leq r \leq 2^m$. Let $R' \subset \{2^m \leq y \leq 2^{m+1}\}$ be such that $R' \cap B \neq \emptyset$. In this case $|R'| \sim |B|$. More precisely, we can show that $\pi/2^5 \leq |B|/|R'| \leq \pi/2$.

In both cases set $c_B = c_6 Q_{I_{R'}}$. We then have

$$\frac{1}{|B|} \int_B |\hat{Q} - Q_{I_{R'}}| dm_2 \leq \frac{1}{|B|} \sum_{\{R \in \mathcal{R} : R \cap B \neq \emptyset\}} c_6 |Q_{I_R} - Q_{I_{R'}}| |R \cap B|.$$

We now use an estimate which follows from [5, p. 223] or equivalently [6, Lemma 2.1], and is given below:

$$|Q_{I_R} - Q_{I_{R'}}| \le c||Q||_* \left(\log |I_{R'}|/|I_R| + \log 2\right). \tag{3.12}$$

This immediately completes the proof in case (1).

In case (2), the number of rectangles R for which $R\cap B \neq \emptyset$ and $|I_{R'}|/|I_R| = 2^k$ for some positive integer k is at most 2^k since $2r \le 2^{m+1}$. Thus we have

$$\begin{aligned}\frac{1}{|B|}\int_B |\hat{Q} - Q_{I_{R'}}| dm_2 &\le \frac{1}{|B|}\sum_{\{R\in\mathcal{R}:R\cap B\neq\emptyset\}} |Q_{I_R} - Q_{I_{R'}}||R\cap B| \\ &\le \frac{1}{|B|}\sum_{\{R\in\mathcal{R}:R\cap B\neq\emptyset\}} c||Q||_*(\log(|I_{R'}|/|I_R| + \log 2)|R| \\ &\le \frac{c||Q||_*}{|B|}\sum_{k\ge 1} 2^k(k+1)(\log 2)\frac{|R'|}{4^k} \\ &\le c||Q||_*\sum_{k\ge 1}(k+1)/2^k = A,\end{aligned}$$

where the third inequality is obtained using the fact that $|I_{R'}|/|I_R| = 2^k$ implies $|R|/|R'| \sim 4^{-k}$ since $|R| \sim |I_R|^2$, and the fourth inequality uses $|R'| \sim |B|$.

This proves the Lemma and the Theorem. ∎

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